

# GROUND STATES OF CRITICAL AND SUPERCRITICAL PROBLEMS OF BREZIS-NIRENBERG TYPE

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ABSTRACT. We study the existence of symmetric ground states to the supercritical problem

$$-\Delta v = \lambda v + |v|^{p-2} v \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega,$$

in a domain of the form

$$\Omega = \{(y, z) \in \mathbb{R}^{k+1} \times \mathbb{R}^{N-k-1} : (|y|, z) \in \Theta\},$$

where  $\Theta$  is a bounded smooth domain such that  $\bar{\Theta} \subset (0, \infty) \times \mathbb{R}^{N-k-1}$ ,  $1 \leq k \leq N-3$ ,  $\lambda \in \mathbb{R}$ , and  $p = \frac{2(N-k)}{N-k-2}$  is the  $(k+1)$ -st critical exponent. We show that symmetric ground states exist for  $\lambda$  in some interval to the left of each symmetric eigenvalue, and that no symmetric ground states exist in some interval  $(-\infty, \lambda_*)$  with  $\lambda_* > 0$  if  $k \geq 2$ .

Related to this question is the existence of ground states to the anisotropic critical problem

$$-\operatorname{div}(a(x)\nabla u) = \lambda b(x)u + c(x)|u|^{2^*-2}u \quad \text{in } \Theta, \quad u = 0 \quad \text{on } \partial\Theta,$$

where  $a, b, c$  are positive continuous functions on  $\bar{\Theta}$ . We give a minimax characterization for the ground states of this problem, study the ground state energy level as a function of  $\lambda$ , and obtain a bifurcation result for ground states.

KEY WORDS: Supercritical elliptic problem, anisotropic critical problem, ground states, bifurcation.

2010 MSC: 35J61 (35J20, 35J25).

## 1. INTRODUCTION

We consider the supercritical Brezis-Nirenberg type problem

$$(\wp_\lambda) \quad \begin{cases} -\Delta v = \lambda v + |v|^{2_{N,k}^*-2} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega$  is given by

$$(1.1) \quad \Omega := \{(y, z) \in \mathbb{R}^{k+1} \times \mathbb{R}^{N-k-1} : (|y|, z) \in \Theta\}$$

for some bounded smooth domain  $\Theta$  in  $\mathbb{R}^{N-k}$  such that  $\bar{\Theta} \subset (0, \infty) \times \mathbb{R}^{N-k-1}$ ,  $1 \leq k \leq N-3$ ,  $\lambda \in \mathbb{R}$ , and  $2_{N,k}^* := \frac{2(N-k)}{N-k-2}$  is the so-called  $(k+1)$ -st critical exponent.

If  $k = 0$  then  $2_{N,0}^* = 2^*$  is the critical Sobolev exponent and problem  $(\wp_\lambda)$  becomes

$$(1.2) \quad \begin{cases} -\Delta v = \lambda v + |v|^{2^*-2} v & \text{in } \Theta, \\ v = 0 & \text{on } \partial\Theta. \end{cases}$$

*Date:* August 20, 2016.

M. Clapp is partially supported by CONACYT grant 237661 and PAPIIT-DGAPA-UNAM grant IN104315 (Mexico), A. Pistoia is partially supported by PRIN 2009-WRJ3W7 grant (Italy).

A celebrated result by Brezis and Nirenberg [1] states that (1.2) has a ground state  $v > 0$  if and only if  $\lambda \in (0, \lambda_1)$  and  $N \geq 4$ , or if  $\lambda \in (\lambda_*, \lambda_1)$  and  $N = 3$ , where  $\lambda_*$  is some number in  $(0, \lambda_1)$ . Moreover, they show that  $\lambda_* = \frac{\lambda_1}{4} > 0$  if  $\Theta$  is a ball. As usual,  $\lambda_m$  denotes the  $m$ -th Dirichlet eigenvalue of  $-\Delta$  in  $\Theta$ .

Problem (1.2) has been widely investigated. Capozzi, Fortunato and Palmieri [2] established the existence of solutions for all  $\lambda > 0$  if  $N \geq 5$  and for all  $\lambda \neq \lambda_m$  if  $N = 4$  (see also [11, 24]). Several multiplicity results are also available, see e.g. [3, 7, 8, 9, 25] and the references therein.

Recently, Szulkin, Weth and Willem [22] gave a minimax characterization for the ground states of problem (1.2) when  $\lambda \geq \lambda_1$ . They established the existence of ground states for  $\lambda \neq \lambda_m$  if  $N = 4$  and for all  $\lambda \geq \lambda_1$  if  $N \geq 5$ .

Concerning the supercritical problem  $(\wp_\lambda)$  with  $k \geq 1$ , Passaseo [16, 17] showed that a nontrivial solution does not exist if  $\lambda = 0$  and  $\Theta$  is a ball. This statement was extended in [5] to more general domains  $\Theta$ , and to some unbounded domains in [6]. On the other hand, existence of multiple solutions has been established in [4, 14, 23].

This work is concerned with the existence of symmetric ground states for the supercritical problem  $(\wp_\lambda)$  with  $k \geq 1$ . Note that the domain  $\Omega$  is invariant under the action of the group  $O(k+1)$  of linear isometries of  $\mathbb{R}^{k+1}$  on the first  $k+1$  coordinates. A function  $v : \Omega \rightarrow \mathbb{R}$  is called  $O(k+1)$ -invariant if  $v(gy, z) = v(y, z)$  for every  $g \in O(k+1)$ ,  $(y, z) \in \mathbb{R}^{k+1} \times \mathbb{R}^{N-k-1}$ . The subspace

$$H_0^1(\Omega)^{O(k+1)} := \{v \in H_0^1(\Omega) : v \text{ is } O(k+1)\text{-invariant}\}$$

of  $H_0^1(\Omega)$  is continuously embedded in  $L^{2_{N,k}^*}(\Omega)$ , so the energy functional  $\mathcal{J}_\lambda : H_0^1(\Omega)^{O(k+1)} \rightarrow \mathbb{R}$  given by

$$\mathcal{J}_\lambda(v) := \frac{1}{2} \int_\Omega |\nabla v|^2 - \frac{\lambda}{2} \int_\Omega v^2 - \frac{1}{2^*} \int_\Omega |v|^{2_{N,k}^*}$$

is well defined. Its critical points are the  $O(k+1)$ -invariant solutions to problem  $(\wp_\lambda)$ . An  $O(k+1)$ -invariant  $(PS)_\tau$ -sequence for  $\mathcal{J}_\lambda$  is a sequence  $(v_k)$  such that

$$v_k \in H_0^1(\Omega)^{O(k+1)}, \quad \mathcal{J}_\lambda(v_k) \rightarrow \tau \quad \text{and} \quad \mathcal{J}_\lambda'(v_k) \rightarrow 0 \text{ in } H^{-1}(\Omega).$$

We set

$$\ell_\lambda^{O(k+1)} := \inf\{\tau > 0 : \text{there exists an } O(k+1)\text{-invariant } (PS)_\tau\text{-sequence for } \mathcal{J}_\lambda\}.$$

This is the lowest possible energy level for a nontrivial  $O(k+1)$ -invariant solution to problem  $(\wp_\lambda)$ . An  $O(k+1)$ -invariant ground state of problem  $(\wp_\lambda)$  is a critical point  $v \in H_0^1(\Omega)^{O(k+1)}$  of  $\mathcal{J}_\lambda$  such that  $\mathcal{J}_\lambda(v) = \ell_\lambda^{O(k+1)}$ . Since  $\mathcal{J}_\lambda$  does not satisfy the Palais-Smale condition, an  $O(k+1)$ -invariant ground state does not necessarily exist.

Let  $0 < \lambda_1^{[k]} < \lambda_2^{[k]} \leq \lambda_3^{[k]} \leq \dots$  be the  $O(k+1)$ -invariant eigenvalues of the problem

$$-\Delta v = \lambda v \quad \text{in } \Omega, \quad v \in H_0^1(\Omega)^{O(k+1)},$$

counted with their multiplicity. Set  $\lambda_0^{[k]} := 0$ . We shall prove the following result for  $O(k+1)$ -invariant ground states.

**Theorem 1.1.** *For every  $1 \leq k \leq N-3$ , the following statements hold true:*

- (a) *Problem  $(\wp_\lambda)$  does not have an  $O(k+1)$ -invariant ground state if  $\lambda \leq 0$ .*

- (b) For each  $m \in \mathbb{N} \cup \{0\}$ , there is a number  $\lambda_{m,*}^{[k]} \in [\lambda_m^{[k]}, \lambda_{m+1}^{[k]})$  with the property that problem  $(\wp_\lambda)$  has an  $O(k+1)$ -invariant ground state for every  $\lambda \in (\lambda_{m,*}^{[k]}, \lambda_{m+1}^{[k]})$  and does not have an  $O(k+1)$ -invariant ground state for any  $\lambda \in [\lambda_m^{[k]}, \lambda_{m,*}^{[k]})$ .
- (c) Let  $\beta := \max\{\text{dist}(x, \{0\} \times \mathbb{R}^{N-k-1}) : x \in \overline{\Theta}\}$ . Then,

$$\lambda_{0,*}^{[k]} \geq \begin{cases} \frac{(k-1)^2}{4\beta^2} & \text{if } 3k \geq N, \\ \frac{k}{(2_{N,k}^* - 1)k - 2_{N,k}^*} & \text{if } 3k \leq N. \end{cases}$$

In particular,  $\lambda_{0,*}^{[k]} > 0$  if  $k \geq 2$ .

This last statement stands in contrast with the case  $k = 0$  where a ground state to problem (1.2) exists for every  $\lambda \in [0, \lambda_1)$  if  $N \geq 4$ . We also show that  $\lambda_{0,*}^{[1]} > 0$  if  $\Theta$  is thin enough, see Proposition 4.4.

As we shall see, the  $O(k+1)$ -invariant ground states of problem  $(\wp_\lambda)$  correspond to the ground states of the critical problem

$$(1.3) \quad -\text{div}(a(x)\nabla u) = \lambda b(x)u + c(x)|u|^{2^*-2}u \quad \text{in } \Theta, \quad u = 0 \quad \text{on } \partial\Theta,$$

with  $2^* = \frac{2n}{n-2}$ ,  $n := \dim \Theta$ ,  $a(x_1, \dots, x_n) = x_1^k$  and  $a = b = c$ .

The critical problem (1.3) with general coefficients  $a \in \mathcal{C}^1(\overline{\Theta})$ ,  $b, c \in \mathcal{C}^0(\overline{\Theta})$  has an interest in its own. We study it in section 2 and give a minimax characterization for its ground states, similar to that in [22]. We study the properties of its ground state energy level as a function of  $\lambda$ , and obtain a bifurcation result for ground states, see Theorem 2.1.

Anisotropic critical problems of the form (1.3) have been studied, for example, by Egnell [10] and, more recently, by Hadiji et al. [12, 13]. They obtained existence and multiplicity results under some assumptions which involve flatness of the coefficient functions at some local maximum or minimum point in the interior of  $\Theta$ . Note that the function  $a(x_1, \dots, x_n) = x_1^k$  attains its minimum on the boundary of  $\Theta$ . This produces a quite different behavior regarding the existence of ground states, as we shall see in the following sections.

Section 2 is devoted to the study of the general anisotropic critical problem. In section 3 we prove a nonexistence result for supercritical problems. It will be used in Section 4 where we prove Theorem 1.1. In the last section we include some questions and remarks.

## 2. GROUND STATES OF THE ANISOTROPIC CRITICAL PROBLEM

In this section we consider the anisotropic Brezis-Nirenberg type problem

$$(2.1) \quad \begin{cases} -\text{div}(a(x)\nabla u) = \lambda b(x)u + c(x)|u|^{2^*-2}u & \text{in } \Theta, \\ u = 0 & \text{on } \partial\Theta, \end{cases}$$

where  $\Theta$  is a bounded smooth domain in  $\mathbb{R}^n$ ,  $n \geq 3$ ,  $\lambda \in \mathbb{R}$ ,  $a \in \mathcal{C}^1(\overline{\Theta})$ ,  $b, c \in \mathcal{C}^0(\overline{\Theta})$  are strictly positive on  $\overline{\Theta}$ , and  $2^* := \frac{2n}{n-2}$  is the critical Sobolev exponent in dimension  $n$ .

We take

$$(2.2) \quad \langle u, v \rangle_a := \int_{\Theta} a(x) \nabla u \cdot \nabla v, \quad \|u\|_a := \left( \int_{\Theta} a(x) |\nabla u|^2 \right)^{1/2},$$

to be the scalar product and the norm in  $H_0^1(\Theta)$ , and

$$|u|_{b,2} := \left( \int_{\Theta} b(x) u^2 \right)^{1/2}, \quad |u|_{c,2^*} := \left( \int_{\Theta} c(x) |u|^{2^*} \right)^{1/2^*},$$

to be the norms in  $L^2(\Theta)$  and  $L^{2^*}(\Theta)$  respectively. They are, clearly, equivalent to the standard ones.

Let  $0 < \lambda_1^{a,b} < \lambda_2^{a,b} \leq \lambda_3^{a,b} \leq \dots$  be the eigenvalues of the problem

$$-\operatorname{div}(a(x)\nabla u) = \lambda b(x)u \quad \text{in } \Theta, \quad u = 0 \quad \text{on } \partial\Theta,$$

counted with their multiplicity, and  $e_1, e_2, e_3, \dots$  be the corresponding normalized eigenfunctions, i.e.  $|e_j|_{b,2} = 1$ . Set

$$Z_0 := \{0\}, \quad Z_m := \operatorname{span}\{e_1, \dots, e_m\},$$

$$Y_m := \{w \in H_0^1(\Theta) : \langle w, z \rangle_a = 0 \text{ for all } z \in Z_m\},$$

$$T_0 := (-\infty, \lambda_1^{a,b}), \quad \text{and} \quad T_m := [\lambda_m^{a,b}, \lambda_{m+1}^{a,b}) \text{ if } m \in \mathbb{N}.$$

The solutions to problem (2.1) are the critical points of the functional  $J_\lambda : H_0^1(\Theta) \rightarrow \mathbb{R}$  given by

$$J_\lambda(u) := \frac{1}{2} \|u\|_a^2 - \frac{\lambda}{2} |u|_{b,2}^2 - \frac{1}{2^*} |u|_{c,2^*}^{2^*}.$$

If  $\lambda \in T_m$  we define

$$\mathcal{N}_\lambda \equiv \mathcal{N}_\lambda(\Theta) := \{u \in H_0^1(\Theta) \setminus Z_m : J'_\lambda(u)u = 0 \text{ and } J'_\lambda(u)z = 0 \text{ for all } z \in Z_m\}.$$

This is a  $\mathcal{C}^1$ -submanifold of codimension  $m+1$  in  $H_0^1(\Theta)$ , cf. [22]. If  $\lambda < \lambda_1^{a,b}$  it is the usual Nehari manifold, and if  $\lambda \geq \lambda_1^{a,b}$  it is the generalized Nehari manifold, introduced by Pankov in [15] and studied by Szulkin and Weth in [20, 21]. Note that  $J'_\lambda(z)z < 0$  for all  $z \in Z_m \setminus \{0\}$ . Clearly, the nontrivial critical points of  $J_\lambda$  belong to  $\mathcal{N}_\lambda$ . Moreover, they coincide with the critical points of its restriction  $J_\lambda|_{\mathcal{N}_\lambda} : \mathcal{N}_\lambda \rightarrow \mathbb{R}$ . The proof of these facts is completely analogous to the one given in [22] for the autonomous case. Set

$$\ell_\lambda \equiv \ell_\lambda^{a,b,c} := \inf_{\mathcal{N}_\lambda} J_\lambda.$$

Following [20] one shows that, for every  $w \in Y_m \setminus \{0\}$ , there exist unique  $t_{\lambda,w} \in (0, \infty)$  and  $z_{\lambda,w} \in Z_m$  such that

$$t_{\lambda,w}w + z_{\lambda,w} \in \mathcal{N}_\lambda,$$

and that

$$J_\lambda(t_{\lambda,w}w + z_{\lambda,w}) = \max_{t>0, z \in Z_m} J_\lambda(tw + z).$$

Let  $\Sigma_m := \{w \in Y_m : \|w\|_a = 1\}$  be the unit sphere in  $Y_m$ . Then,

$$(2.3) \quad \ell_\lambda = \inf_{w \in \Sigma_m} \max_{\substack{t>0, \\ z \in Z_m}} J_\lambda(tw + z).$$

As usual, we denote the best Sobolev constant for the embedding  $H^1(\mathbb{R}^n) \hookrightarrow L^{2^*}(\mathbb{R}^n)$  by  $S$ . We set

$$\kappa^{a,c} := \left( \min_{x \in \Theta} \frac{a(x)^{\frac{n}{2}}}{c(x)^{\frac{n-2}{2}}} \right) \frac{1}{n} S^{\frac{n}{2}},$$

and define

$$\lambda_{m,*}^{a,b,c} := \inf\{\lambda \in T_m : \ell_\lambda < \kappa^{a,c}\}.$$

**Theorem 2.1.** *For every  $m \in \mathbb{N} \cup \{0\}$  the following statements hold true:*

(a) *The function  $\lambda \mapsto \ell_\lambda$  is nonincreasing in  $T_m$  and*

$$0 < \ell_\lambda \leq \kappa^{a,c} \quad \text{for all } \lambda \in T_m.$$

(b)  *$\ell_\lambda$  is attained on  $\mathcal{N}_\lambda$  if  $\ell_\lambda < \kappa^{a,c}$ .*

(c) *The function  $\lambda \mapsto \ell_\lambda$  is continuous in  $T_m$  and*

$$\lim_{\lambda \nearrow \lambda_{m+1}^{a,b}} \ell_\lambda = 0.$$

Hence,  $\lambda_{m,*}^{a,b,c} < \lambda_{m+1}^{a,b}$ .

(d)  *$\ell_\lambda$  is not attained if  $\lambda \in (-\infty, \lambda_{0,*}^{a,b,c})$  or  $\lambda \in [\lambda_m^{a,b}, \lambda_{m,*}^{a,b,c})$ ,  $m \geq 1$ , and is attained if  $\lambda \in (\lambda_{m,*}^{a,b,c}, \lambda_{m+1}^{a,b})$ .*

**Remark 2.2.** It follows from part (c) above that bifurcation (to the left) occurs at each  $\lambda_m^{a,b}$ . This fact is essentially known and can be obtained by other methods. However, we would like to emphasize that here we show that our bifurcating solutions are *ground states*.

*Proof of Theorem 2.1.* (a): Let  $\lambda, \mu \in T_m$ . If  $\lambda \leq \mu$  then  $J_\lambda(u) \geq J_\mu(u)$  for every  $u \in H_0^1(\Theta)$ . So  $\ell_\lambda \geq \ell_\mu$  according to (2.3). This proves that  $\lambda \mapsto \ell_\lambda$  is nonincreasing in  $T_m$ .

If  $\lambda \in T_m$  and  $w \in \Sigma_m$  we have that

$$(2.4) \quad \begin{aligned} \max_{t>0, z \in Z_m} J_\lambda(tw + z) &\geq \max_{t>0} J_\lambda(tw) = \frac{1}{n} \left( \frac{\|w\|_a^2 - \lambda |w|_{b,2}^2}{|w|_{c,2^*}^2} \right)^{n/2} \\ &\geq \frac{1}{n} \left( \frac{1 - \frac{\lambda}{\lambda_{m+1}}}{|w|_{c,2^*}^2} \right)^{n/2}. \end{aligned}$$

Using Sobolev's inequality we conclude that there is a positive constant  $C$  such that

$$\max_{t>0, z \in Z_m} J_\lambda(tw + z) \geq C \quad \text{for all } w \in \Sigma_m.$$

Therefore,  $\ell_\lambda > 0$ .

Let  $\varphi_k \in \mathcal{C}_c^\infty(\mathbb{R}^n)$  be a positive function such that  $\text{supp}(\varphi_k) \subset B_{1/k}(0)$  and  $\int |\nabla \varphi_k|^2 \rightarrow S^{n/2}$ ,  $\int |\varphi_k|^{2^*} \rightarrow S^{n/2}$ , where  $B_r(\xi) = \{x \in \mathbb{R}^n : |x - \xi| < r\}$ . Let  $\xi \in \bar{\Theta}$  be such that

$$\frac{a(\xi)^{\frac{n}{2}}}{c(\xi)^{\frac{n-2}{2}}} = \min_{x \in \bar{\Theta}} \frac{a(x)^{\frac{n}{2}}}{c(x)^{\frac{n-2}{2}}}$$

and choose  $\nu \in \mathbb{R}^n$  with  $|\nu| = 1$  such that  $\nu$  is the inward pointing unit normal at  $\xi$  if  $\xi \in \partial\Theta$ . Set  $\xi_k := \xi + \frac{1}{k}\nu$  and  $u_k(x) := \varphi_k(x - \xi_k)$ . Then  $u_k \in H_0^1(\Theta)$  for  $k$  large

enough, and we have that

$$\begin{aligned}
 (2.5) \quad \max_{t>0} J_\lambda(tu_k) &= \frac{1}{n} \left( \frac{\|u_k\|_a^2 - \lambda |u_k|_{b,2}^2}{|u_k|_{c,2^*}^2} \right)^{\frac{n}{2}} \\
 &= \frac{1}{n} \left( \frac{\int_{B_{1/k}(\xi_k)} a(x) |\nabla u_k|^2 - \lambda \int_{B_{1/k}(\xi_k)} b(x) u_k^2}{\left( \int_{B_{1/k}(\xi_k)} c(x) |u_k|^{2^*} \right)^{2/2^*}} \right)^{\frac{n}{2}} \\
 &\longrightarrow \frac{1}{n} \left( \frac{a(\xi)^{\frac{n}{2}}}{c(\xi)^{\frac{n-2}{2}}} \right) S^{\frac{n}{2}} = \kappa^{a,c} \quad \text{as } k \rightarrow \infty.
 \end{aligned}$$

Hence,  $\ell_\lambda \leq \kappa^{a,c}$  for  $\lambda < \lambda_1^{a,b}$ .

Next, we assume that  $\lambda \in T_m$  with  $m \in \mathbb{N}$ . We fix an open subset  $\theta$  of  $\Theta$  such that  $\theta \cap B_{1/k}(\xi_k) = \emptyset$  for  $k$  large enough. If  $z \in Z_m$  and  $z = 0$  in  $\theta$  then  $z = 0$  in  $\Theta$ , see [22, Lemma 3.3]. Hence,  $(\int_\theta c(x) |z|^{2^*})^{1/2^*}$  is a norm in  $Z_m$  and, since  $Z_m$  is finite-dimensional, this norm is equivalent to  $\|z\|_a$ . In particular, there is a positive constant  $A$  such that  $\int_\theta c(x) |z|^{2^*} \geq 2^* A \|z\|_a^{2^*}$  for all  $z \in Z_m$ . It follows by convexity that, for every  $t > 0$  and every  $z \in Z_m$ , we have

$$\begin{aligned}
 |tu_k + z|_{c,2^*}^{2^*} &= \int_{\Theta \setminus \theta} c(x) |tu_k + z|^{2^*} + \int_\theta c(x) |z|^{2^*} \\
 &\geq t^{2^*} \int_\Theta c(x) u_k^{2^*} + 2^* t^{2^*-1} \int_\Theta c(x) u_k^{2^*-1} z + 2^* A \|z\|_a^{2^*}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 (2.6) \quad J_\lambda(tu_k + z) &\leq J_0(tu_k) - \frac{\lambda}{2} |tu_k|_{b,2}^2 + t \int_\Theta (a(x) \nabla u_k \nabla z - \lambda b(x) u_k z) \\
 &\quad + \frac{1}{2} \left( \|z\|_a^2 - \lambda |z|_{b,2}^2 \right) - t^{2^*-1} \int_\Theta c(x) u_k^{2^*-1} z - A \|z\|_a^{2^*} \\
 &\leq J_0(tu_k) + t \int_\Theta (a(x) \nabla u_k \nabla z - \lambda b(x) u_k z) \\
 &\quad - t^{2^*-1} \int_\Theta c(x) u_k^{2^*-1} z - A \|z\|_a^{2^*}.
 \end{aligned}$$

Consequently,

$$J_\lambda(tu_k + z) \leq B(t^2 + t \|z\|_a + t^{2^*-1} \|z\|_a) - C(t^{2^*} + \|z\|_a^{2^*})$$

for some positive constants  $B$  and  $C$ . This implies that there exists  $R > 0$  such that  $J_\lambda(tu_k + z) \leq 0$  for all  $t \geq R$ ,  $z \in Z_m$  and  $k$  large enough. On the other hand, for  $t \leq R$ ,  $z \in Z_m$  and  $k$  large enough, since  $\varphi_k \rightharpoonup 0$  weakly in  $H_0^1(\Theta)$ , inequalities (2.6) and (2.5) imply that

$$J_\lambda(tu_k + z) \leq J_0(tu_k) + o(1) = \kappa^{a,c} + o(1).$$

This proves that  $\ell_\lambda \leq \kappa^{a,c}$  for  $\lambda \geq \lambda_1^{a,b}$  and concludes the proof of statement (a).

(b): Let  $I_\lambda : \Sigma_m \rightarrow \mathbb{R}$  be the function given by

$$I_\lambda(w) := J_\lambda(t_{\lambda,w} w + z_{\lambda,w}).$$

Then  $\ell_\lambda := \inf_{w \in \Sigma_m} I_\lambda(w)$ . It is shown in [20, 21] that  $I_\lambda \in C^1(\Sigma_m, \mathbb{R})$ . Since  $\Sigma_m$  is a smooth submanifold of  $H_0^1(\Theta)$ , Ekeland's variational principle yields a Palais-Smale sequence  $(w_k)$  for  $I_\lambda$  such that  $I_\lambda(w_k) \rightarrow \ell_\lambda$ , cf. [24, Theorem 8.5]. Set  $u_k := t_{\lambda, w_k} w_k + z_{\lambda, w_k}$ . By Corollary 2.10 in [20] or Corollary 3.3 in [21],  $(u_k)$  is a Palais-Smale sequence for  $J_\lambda$ . Now, Corollary 3.2 in [4] asserts that every Palais-Smale sequence  $(u_k)$  for  $J_\lambda$  such that  $J_\lambda(u_k) \rightarrow \tau < \kappa^{a,c}$ , contains a convergent subsequence. It follows that  $\ell_\lambda$  is attained on  $\mathcal{N}_\lambda$  if  $\ell_\lambda < \kappa^{a,c}$ .

(c): Let  $w \in \Sigma_m$ . First, we will show that the function  $\lambda \mapsto I_\lambda(w)$  is continuous in  $T_m$ . Let  $\mu_j, \mu \in T_m$  be such that  $\mu_j \rightarrow \mu$ . A standard argument shows that  $J_{\mu_j}(tw + z) \leq 0$  for every  $j \in \mathbb{N}$  if  $t^2 + \|z\|_a^2$  is large enough. Therefore, the sequences  $(t_{\mu_j, w})$  and  $(z_{\mu_j, w})$  are bounded and, after passing to a subsequence,  $t_{\mu_j, w} \rightarrow t_0$  in  $[0, \infty)$  and  $z_{\mu_j, w} \rightarrow z_0$  in  $Z_m$ . Hence,

$$I_{\mu_j}(w) = J_{\mu_j}(t_{\mu_j, w}w + z_{\mu_j, w}) \rightarrow J_\mu(t_0w + z_0) \leq I_\mu(w).$$

If  $J_\mu(t_0w + z_0) < I_\mu(w)$  then, since

$$J_{\mu_j}(t_{\mu, w}w + z_{\mu, w}) \rightarrow J_\mu(t_{\mu, w}w + z_{\mu, w}) = I_\mu(w),$$

we would have that, for  $j$  large enough,

$$\max_{t>0, z \in Z_m} J_{\mu_j}(tw + z) = J_{\mu_j}(t_{\mu_j, w}w + z_{\mu_j, w}) < J_{\mu_j}(t_{\mu, w}w + z_{\mu, w}),$$

which is a contradiction. Consequently,  $I_{\mu_j}(w) \rightarrow I_\mu(w)$ . This proves that  $\lambda \mapsto I_\lambda(w)$  is continuous in  $T_m$  for each  $w \in \Sigma_m$ .

Next, we prove that the function  $\lambda \mapsto \ell_\lambda$  is continuous from the left in  $T_m$ . Let  $\mu_j, \mu \in T_m$  be such that  $\mu_j \leq \mu$  and  $\mu_j \rightarrow \mu$ . Since the infimum of any family of continuous functions is upper semicontinuous and  $\lambda \mapsto \ell_\lambda$  is nonincreasing, we have that

$$\limsup_{j \rightarrow \infty} \ell_{\mu_j} \leq \ell_\mu \leq \liminf_{j \rightarrow \infty} \ell_{\mu_j}.$$

This proves that  $\lambda \mapsto \ell_\lambda$  is continuous from the left in  $T_m$ .

To prove that  $\lambda \mapsto \ell_\lambda$  is continuous from the right in  $T_m$  we argue by contradiction. Assume there are  $\mu_j, \mu \in T_m$  such that  $\mu_j \geq \mu$ ,  $\mu_j \rightarrow \mu$  and  $\sup_{j \in \mathbb{N}} \ell_{\mu_j} < \ell_\mu$ . Then  $\ell_{\mu_j} < \kappa^{a,c}$  and, by statement (b), there exists  $w_j \in \Sigma_m$  such that  $\ell_{\mu_j} = J_{\mu_j}(t_{\mu_j, w}w_j + z_{\mu_j, w})$ . Inequality (2.4) asserts that

$$\ell_\mu > \ell_{\mu_j} = J_{\mu_j}(t_{\mu_j, w}w_j + z_{\mu_j, w}) \geq \frac{1}{n} \left( \frac{1 - \frac{\mu_j}{\lambda_{m+1}}}{|w_j|_{c, 2^*}^2} \right)^{n/2}.$$

This implies that  $|w_j|_{c, 2^*}^{2^*} \geq \varepsilon > 0$  for all  $j \in \mathbb{N}$ . Denote the closure of  $Y_m$  in  $L^{2^*}(\Theta)$  by  $\tilde{Y}_m$ . Since  $\dim(Z_m) < \infty$ , the projection  $\tilde{Y}_m \oplus Z_m \rightarrow \tilde{Y}_m$  is continuous in  $L^{2^*}(\Theta)$ . Hence, there is a positive constant  $A_0$  such that

$$\begin{aligned} \ell_\mu &\leq J_\mu(t_{\mu, w}w_j + z_{\mu, w_j}) \\ &= \frac{t_{\mu, w_j}^2}{2} (1 - \mu |w_j|_{b, 2}^2) + \frac{1}{2} (\|z_{\mu, w_j}\|_a^2 - \mu |z_{\mu, w_j}|_{b, 2}^2) - \frac{1}{2^*} |t_{\mu, w_j}w_j + z_{\mu, w_j}|_{c, 2^*}^{2^*} \\ &\leq \frac{t_{\mu, w_j}^2}{2} - A_0 \frac{t_{\mu, w_j}^{2^*}}{2^*} |w_j|_{c, 2^*}^{2^*} \leq \frac{t_{\mu, w_j}^2}{2} - A_0 \varepsilon \frac{t_{\mu, w_j}^{2^*}}{2^*} \quad \text{for all } j \in \mathbb{N}. \end{aligned}$$

It follows that  $(t_{\mu,w_j})$  is bounded. Hence,  $(\|z_{\mu,w_j}\|_a)$  is bounded too. Consequently,

$$\begin{aligned} \ell_\mu &\leq J_\mu(t_{\mu,w_j}w_j + z_{\mu,w_j}) = J_{\mu_j}(t_{\mu,w_j}w_j + z_{\mu,w_j}) + (\mu - \mu_j) |t_{\mu,w_j}w_j + z_{\mu,w_j}|_{b,2}^2 \\ &\leq J_{\mu_j}(t_{\mu_j,w_j}w_j + z_{\mu_j,w_j}) + o(1) = \ell_{\mu_j} + o(1) \leq \sup_{j \in \mathbb{N}} \ell_{\mu_j} + o(1) < \ell_\mu + o(1). \end{aligned}$$

This is a contradiction. It follows that the function  $\lambda \mapsto \ell_\lambda$  is continuous in  $T_m$ .

Finally, let  $\mu_j \in T_m$  be such that  $\mu_j \rightarrow \lambda_{m+1}$ . We have that

$$\begin{aligned} 0 &< \ell_{\mu_j} \leq J_{\mu_j}(t_{\mu_j,e_{m+1}}e_{m+1} + z_{\mu_j,e_{m+1}}) \\ &= \frac{t_{\mu_j,e_{m+1}}^2}{2}(\lambda_{m+1} - \mu_j) + \frac{1}{2}(\|z_{\mu_j,e_{m+1}}\|_a^2 - \mu_j |z_{\mu_j,e_{m+1}}|_{b,2}^2) \\ &\quad - \frac{1}{2^*} |t_{\mu_j,e_{m+1}}e_{m+1} + z_{\mu_j,e_{m+1}}|_{c,2^*}^{2^*} \\ &\leq \frac{t_{\mu_j,e_{m+1}}^2}{2}(\lambda_{m+1} - \mu_j) - A_0 \frac{t_{\mu_j,e_{m+1}}^{2^*}}{2^*} |e_{m+1}|_{c,2^*}^{2^*}. \end{aligned}$$

It follows that  $(t_{\mu_j,e_{m+1}})$  is bounded and, hence, that

$$0 < \ell_{\mu_j} \leq \frac{t_{\mu_j,e_{m+1}}^2}{2}(\lambda_{m+1} - \mu_j) = o(1).$$

This proves that  $\ell_{\mu_j} \rightarrow 0$  as  $\mu_j \rightarrow \lambda_{m+1}$  from the left.

(d): If  $\lambda \in T_m$ ,  $\lambda \leq \lambda_{m,*}^{a,b,c}$ , and  $w \in \Sigma_m$  were such that  $\ell_\lambda = I_\lambda(w)$  then for  $\mu \in (\lambda, \lambda_{m,*}^{a,b,c}(\Theta))$  we would have that

$$\kappa^{a,c} = \ell_\mu \leq I_\mu(w) < I_\lambda(w) = \ell_\lambda,$$

contradicting (a). It follows that  $\ell_\lambda$  is not attained if  $\lambda \in [\lambda_m^{a,b}, \lambda_{m,*}^{a,b,c})$ . Statement (b) implies that  $\ell_\lambda$  is attained if  $\lambda \in (\lambda_{m,*}^{a,b,c}, \lambda_{m+1}^{a,b})$ .  $\square$

Recall that a  $(PS)_\tau$ -sequence for  $J_\lambda$  is a sequence  $(u_k)$  in  $H_0^1(\Theta)$  such that  $J_\lambda(u_k) \rightarrow \tau$  and  $J'_\lambda(u_k) \rightarrow 0$  in  $H^{-1}(\Theta)$ . The value  $\ell_\lambda$  is characterized as follows.

**Corollary 2.3.**  $\ell_\lambda = \inf\{\tau > 0 : \text{there exists a } (PS)_\tau\text{-sequence for } J_\lambda\}$ .

*Proof.* The argument given in the proof of statement (b) of Theorem 2.1 shows that there exists a  $(PS)_{\ell_\lambda}$ -sequence for  $J_\lambda$ . To prove that  $\ell_\lambda$  is the smallest positive number with this property, we argue by contradiction. Assume that  $\tau < \ell_\lambda$  and that there exists a  $(PS)_\tau$ -sequence for  $J_\lambda$ . Then  $\tau < \kappa^{a,c}$  and Corollary 3.2 in [4] asserts that  $(u_k)$  contains a subsequence which converges to a critical point  $u$  of  $J_\lambda$  with  $J_\lambda(u) = \tau$ . If  $\tau \neq 0$  then  $u \in \mathcal{N}_\lambda$  and, hence,  $\ell_\lambda \leq \tau$ . This is a contradiction.  $\square$

For the classical Brezis-Nirenberg problem (1.2) (where  $a = b = c \equiv 1$ ) with  $n \geq 4$ , it is known that  $\lambda_{0,*}^{a,b,c} = 0$  and  $\lambda_{m,*}^{a,b,c} = \lambda_m$ , the  $m$ -th Dirichlet eigenvalue of  $-\Delta$  in  $\Theta$ , for all  $m \in \mathbb{N}$ . Moreover,  $\ell_\lambda = \frac{1}{n} S^{\frac{n}{2}} = \kappa^{a,c}$  for every  $\lambda \leq 0$ , but  $\ell_\lambda < \frac{1}{n} S^{\frac{n}{2}}$  for every  $\lambda > 0$  if  $n \geq 5$ , see [1, 11, 22].

As we shall see below, this is not true in general: For the problem  $(\wp_\lambda^\#)$  in Section 4 which arises from the supercritical one, one has that  $\lambda_{0,*}^{a,b,c} > 0$  in most cases, see Propositions 4.3 and 4.4. A special feature of that problem is that the value  $\kappa^{a,c}$  is attained on the boundary of  $\Theta$ . A different situation was considered by Egnell [10] and Hadiji and Yazidi [13]. They showed for example that, if  $a$  attains its minimum at an interior point  $x_0$  of  $\Theta$ ,  $b = 1 = c$ , and  $a$  is flat enough around  $x_0$ , then  $\lambda_{0,*}^{a,b,c} = 0$  for  $n \geq 4$ , as in the classical Brezis-Nirenberg case.



We do not know whether, in general,  $\lambda_{0,*}^{a,b,c} \geq 0$ . But this will be true in the special case we are interested in, see Proposition 4.1. The proof uses a nonexistence result for the supercritical problem, which we discuss in the following section.

### 3. NONEXISTENCE OF SOLUTIONS TO A SUPERCRITICAL PROBLEM

Let  $\Theta$  be a bounded smooth domain in  $\mathbb{R}^{N-k}$  with  $\overline{\Theta} \subset (0, \infty) \times \mathbb{R}^{N-k-1}$  and  $0 \leq k \leq N-3$ . Set

$$\Omega := \{(y, z) \in \mathbb{R}^{k+1} \times \mathbb{R}^{N-k-1} : (|y|, z) \in \Theta\}$$

and consider the problem

$$(3.1) \quad \begin{cases} -\Delta u = \lambda u + |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Passaseo [16, 17] showed that, if  $\Theta$  is a ball, problem (3.1) does not have a nontrivial solution for  $\lambda = 0$  and  $p \geq 2_{N,k}^* := \frac{2(N-k)}{N-k-2}$ . In [5] it is shown that this is also true for doubly starshaped domains.

**Definition 3.1.**  $\Theta$  is doubly starshaped if there exist two numbers  $0 < t_0 < t_1$  such that  $t \in (t_0, t_1)$  for every  $(t, z) \in \Theta$  and  $\Theta$  is strictly starshaped with respect to  $\xi_0 := (t_0, 0)$  and to  $\xi_1 := (t_1, 0)$ , i.e.

$$\langle x - \xi_i, \nu_\Theta(x) \rangle > 0 \quad \forall x \in \partial\Theta \setminus \{\xi_i\}, \quad i = 0, 1,$$

where  $\nu_\Theta$  is the outward pointing unit normal to  $\partial\Theta$ .

We denote the first Dirichlet eigenvalue of  $-\Delta$  in  $\Omega$  by  $\lambda_1(\Omega)$ .

**Theorem 3.2.** If  $\Theta$  is doubly starshaped,  $p \geq 2_{N,k}^*$  and

$$\lambda \leq \frac{2(p - 2_{N,k}^*)}{2_{N,k}^*(p - 2)} \lambda_1(\Omega),$$

then problem (3.1) does not have a nontrivial solution.

We point out that the geometric assumption on  $\Theta$  cannot be dropped. Existence of multiple solutions to problem (3.1) for  $\lambda = 0$  and  $p = 2_{N,k}^*$  in some domains where  $\Theta$  is not doubly starshaped has been established in [4, 14, 23].

The proof of Theorem 3.2 follows the ideas introduced in [5, 16, 17]. Fix  $\tau \in (0, \infty)$  and let  $\varphi$  be the solution to the problem

$$\begin{cases} \varphi'(t)t + (k+1)\varphi(t) = 1, & t \in (0, \infty), \\ \varphi(\tau) = 0. \end{cases}$$

Explicitly,  $\varphi(t) = \frac{1}{k+1} [1 - (\frac{\tau}{t})^{k+1}]$ . Note that  $\varphi$  is strictly increasing in  $(0, \infty)$ . For  $y \neq 0$  we define

$$(3.2) \quad \chi_\tau(y, z) := (\varphi(|y|)y, z).$$

**Lemma 3.3.** The vector field  $\chi_\tau$  has the following properties:

- (a)  $\operatorname{div} \chi_\tau = N - k$ ,
- (b)  $\langle d\chi_\tau(y, z)[\xi], \xi \rangle \leq \max\{1 - k\varphi(|y|), 1\} |\xi|^2$  for every  $y \in \mathbb{R}^{k+1} \setminus \{0\}$ ,  $z \in \mathbb{R}^{N-k-1}$ ,  $\xi \in \mathbb{R}^N$ .

*Proof.* See [17, Lemma 2.3] or [5, Lemma 4.2]. □

**Proposition 3.4.** *Assume there exists  $\tau \in (0, \infty)$  such that  $|y| \in (\tau, \infty)$  for every  $(y, z) \in \Omega$  and  $\langle \chi_\tau, \nu_\Omega \rangle > 0$  a.e. on  $\partial\Omega$ . If  $p \geq 2_{N,k}^*$  and*

$$\lambda \leq \frac{2(p - 2_{N,k}^*)}{2_{N,k}^*(p - 2)} \lambda_1(\Omega),$$

*then problem (3.1) does not have a nontrivial solution.*

*Proof.* The variational identity (4) in Pucci and Serrin's paper [19] implies that, if  $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$  is a solution of (3.1) and  $\chi \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R}^N)$ , then

$$(3.3) \quad \begin{aligned} \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 \langle \chi, \nu_\Omega \rangle d\sigma &= \int_{\Omega} (\operatorname{div} \chi) \left[ \frac{1}{p} |u|^p + \frac{\lambda}{2} u^2 - \frac{1}{2} |\nabla u|^2 \right] dx \\ &\quad + \int_{\Omega} \langle d\chi [\nabla u], \nabla u \rangle dx, \end{aligned}$$

where  $\nu_\Omega$  is the outward pointing unit normal to  $\partial\Omega$  (in the notation of [19] we have taken  $\mathcal{F}(x, u, \nabla u) = \frac{1}{2} |\nabla u|^2 - \frac{1}{2} \lambda u^2 - \frac{1}{p} |u|^p$ ,  $h = \chi$  and  $a = 0$ ). Let  $\chi := \chi_\tau$ . Then, by Lemma 3.3,

$$\operatorname{div} \chi_\tau = N - k.$$

Moreover, since  $1 - k\varphi(t) < 1$  for  $t \in (\tau, \infty)$ , and  $|y| \in (\tau, \infty)$  for every  $(y, z) \in \Omega$ , Lemma 3.3 yields

$$\langle d\chi_\tau(y, z) [\xi], \xi \rangle \leq |\xi|^2 \quad \forall (y, z) \in \Omega, \xi \in \mathbb{R}^N.$$

By assumption,  $\langle \chi_\tau, \nu_\Omega \rangle > 0$  a.e. on  $\partial\Omega$ . Therefore, if  $u$  is a nontrivial solution of (3.1) we have, using (3.3), that

$$\begin{aligned} 0 &< (N - k) \left( \frac{1}{p} - \frac{1}{2} \right) \int_{\Omega} [|\nabla u|^2 - \lambda u^2] dx + \int_{\Omega} |\nabla u|^2 dx \\ &= (N - k) \left( \frac{1}{p} - \frac{1}{2} + \frac{1}{N - k} \right) \int_{\Omega} |\nabla u|^2 dx - (N - k) \left( \frac{1}{p} - \frac{1}{2} \right) \lambda \int_{\Omega} u^2 dx, \end{aligned}$$

that is,

$$\left( \frac{1}{2} - \frac{1}{p} \right) \lambda \int_{\Omega} u^2 dx > \left( \frac{1}{2_{N,k}^*} - \frac{1}{p} \right) \int_{\Omega} |\nabla u|^2 dx.$$

Therefore, if  $p \geq 2_{N,k}^*$  and

$$\lambda \leq \frac{2(p - 2_{N,k}^*)}{2_{N,k}^*(p - 2)} \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} u^2 dx} = \frac{2(p - 2_{N,k}^*)}{2_{N,k}^*(p - 2)} \lambda_1(\Omega),$$

problem (3.1) does not have a nontrivial solution in  $\Omega$ , as claimed.  $\square$

The following result was proved in [5].

**Proposition 3.5.** *If  $\Theta$  is doubly starshaped then  $\Theta \subset (t_0, \infty) \times \mathbb{R}^{N-k-1}$  and  $\langle \chi_{t_0}, \nu_\Theta \rangle > 0$  a.e. on  $\partial\Theta$ , with  $t_0$  as in Definition 3.1.*

*Proof.* See the proof of (4.11) in [5].  $\square$

*Proof of Theorem 3.2.* The conclusion follows immediately from Propositions 3.4 and 3.5.  $\square$

4. EXISTENCE AND NONEXISTENCE OF SYMMETRIC GROUND STATES TO  
 SUPERCRITICAL PROBLEMS

Next, we come back to our original supercritical problem

$$(\wp_\lambda) \quad \begin{cases} -\Delta v = \lambda v + |v|^{2_{N,k}^*-2} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\Omega := \{(y, z) \in \mathbb{R}^{k+1} \times \mathbb{R}^{N-k-1} : (|y|, z) \in \Theta\}$$

for some bounded smooth domain  $\Theta$  in  $\mathbb{R}^{N-k}$  with  $\bar{\Theta} \subset (0, \infty) \times \mathbb{R}^{N-k-1}$ ,  $1 \leq k \leq N-3$ , and  $2_{N,k}^* := \frac{2(N-k)}{N-k-2}$ .

An  $O(k+1)$ -invariant function  $v : \Omega \rightarrow \mathbb{R}$  can be written as  $v(y, z) = u(|y|, z)$  for some function  $u : \Theta \rightarrow \mathbb{R}$ . A straightforward computation shows that

$$(4.1) \quad \Delta v = \frac{1}{a(x)} \operatorname{div}(a(x) \nabla u),$$

where  $a(x_1, \dots, x_{N-k}) := x_1^k$ . Hence,  $v$  is an  $O(k+1)$ -invariant solution of  $(\wp_\lambda)$  if and only if  $u$  solves

$$(\wp_\lambda^\#) \quad \begin{cases} -\operatorname{div}(x_1^k \nabla u) = \lambda x_1^k u + x_1^k |u|^{2^*-2} u & \text{in } \Theta, \\ u = 0 & \text{on } \partial\Theta, \end{cases}$$

where  $2^* = 2_{N,k}^*$  is the critical exponent in dimension  $n := N-k = \dim(\Theta)$ . So this problem is a special case of the problem treated in section 2 with  $a(x_1, \dots, x_n) := x_1^k$  and  $a = b = c$ .

For these functions  $a, b, c$  we simplify notation and write  $\ell_\lambda^{[k]}$ ,  $\kappa^{[k]}$ ,  $\lambda_m^{[k]}$ ,  $\lambda_{m,*}^{[k]}$  instead of  $\ell_\lambda^{a,b,c}$ ,  $\kappa^{a,c}$ ,  $\lambda_m^{a,b}$ ,  $\lambda_{m,*}^{a,b,c}$ . Note that

$$\kappa^{[k]} = \left( \min_{x \in \bar{\Theta}} x_1^k \right) \frac{1}{n} S^{n/2}.$$

**Proposition 4.1.** *If  $\alpha := \min_{x \in \bar{\Theta}} x_1$  and  $\lambda \leq 0$ , then  $\ell_\lambda^{[k]} = \frac{\alpha^k}{n} S^{n/2}$  and it is not attained by  $J_\lambda$  on  $\mathcal{N}_\lambda \equiv \mathcal{N}_\lambda(\Theta)$ .*

*Proof.* By Theorem 2.1 it is enough to show this for  $\lambda = 0$ . Arguing by contradiction, assume that  $\ell_0^{[k]} < \frac{\alpha^k}{n} S^{n/2}$ . Then there exists  $\varphi \in \mathcal{C}_c^\infty(\Theta) \cap \mathcal{N}_0(\Theta)$  such that

$$J_0(\varphi) < \frac{\alpha^k}{n} S^{n/2} = \kappa^{[k]}.$$

Since  $\operatorname{supp}(\varphi)$  is a compact subset of  $(\alpha, \infty) \times \mathbb{R}^{n-1}$ , there exists a  $\varrho \in (\alpha, \infty)$  such that  $\operatorname{supp}(\varphi) \subset B := \{x \in \mathbb{R}^n : (x_1 - \varrho)^2 + x_2^2 + \dots + x_n^2 < (\alpha - \varrho)^2\}$ . Hence,  $\varphi \in \mathcal{N}_0(B)$ . Theorem 3.2 and the discussion given at the beginning of this section imply that problem

$$-\operatorname{div}(x_1^k \nabla u) = x_1^k |u|^{2^*-2} u \quad \text{in } B, \quad u = 0 \quad \text{on } \partial B$$

does not have a nontrivial solution. So, by Theorem 2.1,  $\inf_{\mathcal{N}_0(B)} J_0 = \kappa^{[k]} = \frac{\alpha^k}{n} S^{n/2}$ . But

$$\inf_{\mathcal{N}_0(B)} J_0 \leq J_0(\varphi) < \frac{\alpha^k}{n} S^{n/2}.$$

This is a contradiction. We conclude that  $\ell_0^{[k]} = \frac{\alpha^k}{n} S^{n/2}$ .

Since this value is the same for every  $\Theta$  such that  $\alpha := \min_{x \in \overline{\Theta}} x_1$ , a standard argument shows that  $\ell_0^{[k]}$  is not attained by  $J_0$  on  $\mathcal{N}_0 \equiv \mathcal{N}_0(\Theta)$ .  $\square$

Set

$$\alpha := \min_{x \in \overline{\Theta}} x_1, \quad \beta := \max_{x \in \overline{\Theta}} x_1.$$

**Lemma 4.2.** *For every positive function  $f \in C^2[\alpha, \beta]$  which satisfies*

$$(4.2) \quad \alpha^k f^2(\alpha) \leq t^k f^2(t) \quad \text{and} \quad t^k f^{2^*}(t) \leq \alpha^k f^{2^*}(\alpha) \quad \forall t \in [\alpha, \beta],$$

*we have that*

$$\lambda_{0,*}^{[k]} \geq \min_{t \in [\alpha, \beta]} \frac{-(t^k f'(t))'}{t^k f(t)}.$$

*Proof.* Let  $u \in H_0^1(\Theta)$ ,  $u \neq 0$ , and set  $u(x) = f(x_1)w(x)$ . Then

$$\begin{aligned} \int_{\Theta} x_1^k |\nabla u|^2 &= \int_{\Theta} \left( x_1^k f^2 |\nabla w|^2 + x_1^k f' f \frac{\partial w^2}{\partial x_1} + x_1^k (f')^2 w^2 \right) \\ &= \int_{\Theta} \left( x_1^k f^2 |\nabla w|^2 + x_1^k f' \frac{\partial(f w^2)}{\partial x_1} \right) \\ &= \int_{\Theta} \left( x_1^k f^2 |\nabla w|^2 - (x_1^k f')' f w^2 \right). \end{aligned}$$

So, if  $\lambda \leq \frac{-(t^k f'(t))'}{t^k f(t)}$  for all  $t \in [\alpha, \beta]$ , we have that

$$\begin{aligned} \frac{\int_{\Theta} x_1^k |\nabla u|^2 - \lambda \int_{\Theta} x_1^k u^2}{\left( \int_{\Theta} x_1^k |u|^{2^*} \right)^{2/2^*}} &= \frac{\int_{\Theta} x_1^k f^2 |\nabla w|^2 - \int_{\Theta} \left[ (x_1^k f')' + \lambda x_1^k f \right] f w^2}{\left( \int_{\Theta} x_1^k f^{2^*} |w|^{2^*} \right)^{2/2^*}} \\ &\geq \frac{\int_{\Theta} x_1^k f^2 |\nabla w|^2}{\left( \int_{\Theta} x_1^k f^{2^*} |w|^{2^*} \right)^{2/2^*}} \geq \frac{\alpha^k f^2(\alpha) \int_{\Theta} |\nabla w|^2}{\alpha^{2k/2^*} f^2(\alpha) \left( \int_{\Theta} |w|^{2^*} \right)^{2/2^*}} \\ &= \alpha^{2k/n} \frac{\int_{\Theta} |\nabla w|^2}{\left( \int_{\Theta} |w|^{2^*} \right)^{2/2^*}} \geq \alpha^{2k/n} S > 0 \quad \text{for all } u \in H_0^1(\Theta), u \neq 0. \end{aligned}$$

This implies that  $\lambda < \lambda_1^{[k]}$  and, hence, that

$$\max_{t>0} J_{\lambda}(tu) = \frac{1}{n} \left( \frac{\int_{\Theta} x_1^k |\nabla u|^2 - \lambda \int_{\Theta} x_1^k u^2}{\left( \int_{\Theta} x_1^k |u|^{2^*} \right)^{2/2^*}} \right)^{n/2} \geq \frac{\alpha^k}{n} S^{n/2}$$

for all  $u \in H_0^1(\Theta)$ ,  $u \neq 0$ . Therefore,  $\ell_{\lambda}^{[k]} = \frac{\alpha^k}{n} S^{n/2}$  for every  $\lambda \leq \min_{t \in [\alpha, \beta]} \frac{-(t^k f'(t))'}{t^k f(t)}$ , and the conclusion follows.  $\square$

We obtain the following estimates for  $\lambda_{0,*}^{[k]}$ .

**Proposition 4.3.**  $\lambda_{0,*}^{[k]} \geq 0$  and

$$\lambda_{0,*}^{[k]} \geq \begin{cases} \frac{(k-1)^2}{4\beta^2} & \text{if } 2k \geq n, \\ \frac{k}{(2^*\beta)^2} ((2^* - 1)k - 2^*) & \text{if } 2k \leq n. \end{cases}$$

Therefore  $\lambda_{0,*}^{[k]} > 0$  if  $k \geq 2$ .

*Proof.* Proposition 4.1 implies that  $\lambda_{0,*}^{[k]} \geq 0$ .

Set  $f(t) := t^{-\gamma}$  with  $\frac{k}{2^*} \leq \gamma \leq \frac{k}{2}$ . This function satisfies (4.2) and, since

$$\frac{-(t^k f'(t))'}{t^k f(t)} = \frac{\gamma(k - \gamma - 1)}{t^2},$$

Lemma 4.2 implies that

$$\lambda_{0,*}^{[k]} \geq \frac{\gamma(k - \gamma - 1)}{\beta^2}.$$

Now observe that the function  $\phi(\gamma) := \gamma(k - \gamma - 1)$  attains its maximum on the interval  $[\frac{k}{2^*}, \frac{k}{2}]$  at the point

$$\gamma_* := \max \left\{ \frac{k-1}{2}, \frac{k}{2^*} \right\}.$$

Therefore  $\lambda_{0,*}^{[k]} \geq \frac{1}{\beta^2} \phi(\gamma_*) = \frac{1}{\beta^2} \gamma_*(k - \gamma_* - 1)$ , as claimed.

Finally, note that  $k > \frac{2^*}{2^*-1} = \frac{2n}{n+2}$  if  $k \geq 2$ . Hence,  $\lambda_{0,*}^{[k]} > 0$  if  $k \geq 2$ .  $\square$

*Proof of Theorem 1.1.* Using Corollary 2.3 it is easily seen that, if  $v(y, z) = u(|y|, z)$ , then  $v$  is an  $O(k+1)$ -invariant ground state for problem  $(\wp_\lambda)$  if and only if  $u$  is a ground state for problem  $(\wp_\lambda^\#)$ . So Theorem 1.1 follows immediately from Proposition 4.1, Theorem 2.1 and Proposition 4.3.  $\square$

The following result shows that  $\lambda_{0,*}^{[1]} > 0$  if the domain is thin enough in the  $x_1$ -direction.

**Proposition 4.4.** *If  $\frac{\beta}{\alpha} \leq \frac{n}{n-2}$  then  $\lambda_{0,*}^{[k]} \geq \frac{k^2}{4\beta^2} > 0$  for all  $k \geq 1$ .*

*Proof.* Set  $f(t) := e^{-\gamma(t-\alpha)}$  with  $\frac{k}{2^*\alpha} \leq \gamma \leq \frac{k}{2\beta}$ , and write  $g(t) := t^k f^2(t)$  and  $h(t) := t^k f^{2^*}(t)$ . Then  $g'(t) = t^{k-1} e^{-2\gamma(t-\alpha)}(k - 2\gamma t) \geq 0$  and  $h'(t) = t^{k-1} e^{-2^*\gamma(t-\alpha)}(k - 2^*\gamma t) \leq 0$  for all  $t \in [\alpha, \beta]$ , so  $f$  satisfies (4.2). Since

$$\frac{-(t^k f'(t))'}{t^k f(t)} = \frac{\gamma t^{k-1} e^{-\gamma(t-\alpha)}(k - \gamma t)}{t^k e^{-\gamma(t-\alpha)}} = \frac{\gamma(k - \gamma t)}{t},$$

Lemma 4.2 implies that

$$\lambda_{0,*}^{[k]} \geq \frac{\gamma(k - \gamma\beta)}{\beta}.$$

Now observe that the function  $\phi(\gamma) := \gamma(k - \gamma\beta)$  attains its maximum at the point  $\gamma_* := \frac{k}{2\beta}$ . Hence,  $\lambda_{0,*}^{[k]} \geq \frac{k^2}{4\beta^2} > 0$ , as claimed.  $\square$

## 5. SOME OPEN QUESTIONS AND COMMENTS

Many questions remain open. Here are some of them.

**Problem 1.** *Concerning problem  $(\wp_\lambda^\#)$ :*

- (1) *Is it true that  $\lambda_{0,*}^{[1]} > 0$  for any domain  $\Theta$ , and not only for thin domains?*
- (2) *For  $m \geq 1$ , is  $\lambda_{m,*}^{[k]} > \lambda_m^{[k]}$ , or is  $\lambda_{m,*}^{[k]} = \lambda_m^{[k]}$  as in the classical Brezis-Nirenberg case?*

- (3) What happens in general at  $\lambda_{m,*}^{[k]}$ ? Is there, or not, a ground state of problem  $(\wp_\lambda^\#)$  for  $\lambda = \lambda_{m,*}^{[k]}$ ?

**Problem 2.** Concerning the general anisotropic problem (2.1):

- (1) Is  $\lambda_{0,*}^{a,b,c}$  always nonnegative? Or are there examples where a ground state exists for some  $\lambda < 0$ ? For all  $\lambda < 0$ ?
- (2) Can one give lower estimates for  $\lambda_{m,*}^{a,b,c}$  in some cases?
- (3) Suppose that  $c \in C^1(\bar{\Theta})$  in addition to our earlier assumptions. If  $\kappa^{a,c}$  is attained only at points which are non-stationary for  $\frac{a(x)^{n/2}}{c(x)^{(n-2)/2}}$  and lie on the boundary of  $\Theta$ , is it then true that  $\lambda_{0,*}^{a,b,c} > 0$ ?

Two particular cases of (3) are:  $c = 1$ , and  $a = b = c$ . If the answer is positive in the first case, this would be in contrast to the results in [10] and [13]. A positive answer in the second case would be a generalization of our results for  $(\wp_\lambda^\#)$ . A partial answer can be given using Proposition 4.3. Consider, for example, the problem

$$(5.1) \quad -\operatorname{div}(a(x)\nabla u) = \lambda b(x)u + |u|^{2^*-2}u \quad \text{in } \Theta, \quad u = 0 \quad \text{on } \partial\Theta,$$

where  $\Theta$  is a bounded smooth domain in  $\mathbb{R}^n$ ,  $n \geq 3$ ,  $\lambda \in \mathbb{R}$ ,  $a \in C^1(\bar{\Theta})$ ,  $b \in C^0(\bar{\Theta})$  are strictly positive on  $\bar{\Theta}$ , and  $2^* = \frac{2n}{n-2}$ . Then, the following statement holds true.

**Proposition 5.1.** *If  $a(x) \geq x_1^k \geq b(x)$  for all  $x \in \Theta$  and  $\min_{x \in \bar{\Theta}} a(x) = (\min_{x \in \bar{\Theta}} x_1)^k > 0$  for some  $k \geq 2$ , then  $\lambda_{0,*}^{a,b,1} > 0$ .*

*Proof.* Let  $\alpha := \min_{x \in \bar{\Theta}} x_1 > 0$ . For every  $u \in H_0^1(\Theta)$ ,  $u \neq 0$ ,  $\lambda \in [0, \lambda_{0,*}^{[k]}]$  we have that

$$\frac{\int_{\Theta} a(x) |\nabla u|^2 - \lambda \int_{\Theta} b(x) u^2}{\alpha^{2k/2^*} \left( \int_{\Theta} |u|^{2^*} \right)^{2/2^*}} \geq \frac{\int_{\Theta} x_1^k |\nabla u|^2 - \lambda \int_{\Theta} x_1^k u^2}{\left( \int_{\Theta} x_1^k |u|^{2^*} \right)^{2/2^*}} > 0.$$

Hence,  $\lambda < \lambda_1^{a,b}$  and

$$\begin{aligned} \frac{1}{n} \left( \frac{\int_{\Theta} a(x) |\nabla u|^2 - \lambda \int_{\Theta} b(x) u^2}{\left( \int_{\Theta} |u|^{2^*} \right)^{2/2^*}} \right)^{n/2} &\geq \alpha^{\frac{k(n-2)}{2}} \frac{1}{n} \left( \frac{\int_{\Theta} x_1^k |\nabla u|^2 - \lambda \int_{\Theta} x_1^k u^2}{\left( \int_{\Theta} x_1^k |u|^{2^*} \right)^{2/2^*}} \right)^{n/2} \\ &\geq \alpha^{\frac{k(n-2)}{2}} \frac{\alpha^k}{n} S^{n/2} = (\alpha^k)^{n/2} \frac{1}{n} S^{n/2} = \kappa^{a,1}. \end{aligned}$$

It follows from Theorem 2.1 that

$$\ell_\lambda^{a,b,1} = \kappa^{a,1} \quad \text{for all } \lambda \in (-\infty, \lambda_{0,*}^{[k]}].$$

Hence, by Proposition 4.3,  $\lambda_{0,*}^{a,b,1} \geq \lambda_{0,*}^{[k]} > 0$ , as claimed.  $\square$

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